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# Bounds on the support of the multifractal spectrum of stochastic processes

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**Abstract:** The multifractal analysis of stochastic processes deals with the fine scale properties of the sample paths and seeks for some global scaling property that would enable extracting the so-called spectrum of singularities. In this paper we establish bounds on the support of the spectrum of singularities. To do this, we prove a theorem that complements the famous Kolmogorov's continuity criterion. The nature of these bounds helps us identify the quantities truly responsible for the support of the spectrum. We then make several conclusions from this. First, specifying global scaling in terms of moments is incomplete due to possible infinite moments, both of positive and negative order. The divergence of negative order moments does not affect the spectrum in general. On the other hand, infinite positive order moments make the spectrum of self-similar processes nontrivial. In particular, we show that the self-similar stationary increments process with the nontrivial spectrum must be heavy-tailed. This shows that for determining the spectrum it is crucial to capture the divergence of moments. We show that the partition function is capable of doing this and also propose a robust variant of this method for negative order moments.

**Keywords:** multifractal, self-similar, spectrum of singularities

# 1 INTRODUCTION

The notion of multifractality first appeared in the setting of measures. The importance of scaling relations was first stressed in the work of Mandelbrot in the context of turbulence modeling ([1, 2]). Later the notion has been extended to functions and studying fine scale properties of functions (see [3, 4, 5]). In this setting, multifractal analysis deals with the local scaling properties of functions characterized by the Hausdorff dimension of sets of points having the same Hölder exponent. The Hausdorff dimension of these sets for varying Hölder exponent yields the so-called spectrum of singularities (or multifractal spectrum). The function is called multifractal if its spectrum is nontrivial, in the sense that it is not a one point set.

However, from a practical point of view, it is impossible to numerically determine the spectrum directly from the definition. Frisch and Parisi ([6]) were the first to propose the idea of determining the spectrum based on certain average quantities as a numerically attainable way. In order to relate this global scaling property and the local one based on the Hölder exponents, one needs “multifractal formalism” to hold. This is not always the case and there has been an extensive research on this topic (see [4, 7, 8, 9, 10]). In order to overcome the problem, one takes the other way around and seeks for different definitions of global and local scaling properties that would always be related by a certain type of multifractal formalism (see [11] for an overview in the context of measures and functions). Many authors claim that wavelets provide the best way to specify the multifractal formalism, both theoretically and numerically (see e.g. [11, 12]).

For stochastic processes, the local scaling properties can be immediately generalized by simply applying the definition for a function on the sample paths. As a global property, the extension is not so straightforward. In [13], the authors present a theory of multifractal stochastic processes and define the scaling property in terms of the moments of the process. The underlying idea is to define a scaling property more general than the well known self-

similarity. However, there are certain discrepancies in the terminology. For example,  $\alpha$ -stable Lévy processes with  $0 < \alpha < 2$  are known to be self-similar with index  $1/\alpha$ . On the other hand, these processes are multifractal from the sample paths point of view, since it follows from [14] that they have a nontrivial spectrum.

The goal of this paper is to make a contribution to the multifractal theory of stochastic processes by exhibiting limitations of the existing definitions and proposing methods to overcome these. The issue of infinite moments has so far been discussed mostly as a problem of the estimation methods for determining the spectrum and has been a major critic for the partition function method. To our best knowledge, our results are the first that link heavy-tails of self-similar processes with their path irregularities in this sense. We illustrate on examples that in this case, ignorant estimation of infinite moments will yield the correct spectrum. Although these bounds are very general, we later restrict our attention to stationary increments processes. We consider only  $\mathbb{R}$ -valued stochastic processes and our treatment is intended to be probabilistic.

The paper is organized as follows. In the next section we review different definitions of multifractal stochastic processes and recall some implications between them. We also discuss the multifractal formalism and different methods for the estimation of spectrum. In Sec. 3 we derive general bounds that determine the support of the multifractal spectrum and relate the bounds with the moment scaling properties. We show implications of these results for self-similar stationary increments processes. Sec. 4 provides examples of stochastic processes from the perspective of different definitions. We show how the results of Sec. 3 apply for each example. In Sec. 5 we propose a simple modification of the partition function method that overcomes divergencies of negative order moments. We illustrate on the simulated data the advantages of this modification. Appendix contains some general facts about processes considered in Sec. 4.

## 2 MULTIFRACTAL STOCHASTIC PROCESSES

In this section we provide an overview of different scaling relations that are usually referred to as multifractality. Examples of processes that satisfy these properties are given in Sec. 4.

The best known scaling relation in the theory of stochastic processes is self-similarity. A stochastic process  $\{X(t), t \geq 0\}$  is said to be self-similar if for any  $a > 0$ , there exists  $b > 0$  such that

$$\{X(at)\} \stackrel{d}{=} \{bX(t)\}, \quad (1)$$

where  $\{\cdot\} \stackrel{d}{=} \{\cdot\}$  stands for the equality of finite dimensional distributions. A process  $\{X(t), t \geq 0\}$  is said to be stochastically continuous at 0 if for every  $\varepsilon > 0$ ,  $P(|X(h) - X(0)| > \varepsilon) \rightarrow 0$  as  $h \rightarrow 0$ . If  $\{X(t), t \geq 0\}$  is self-similar, nontrivial (in the sense that it is not a.s. constant) and stochastically continuous at 0, then  $b$  in (1) must be of the form  $a^H$  for some  $H \geq 0$ , i.e.

$$\{X(at)\} \stackrel{d}{=} \{a^H X(t)\}. \quad (2)$$

The proof of this fact can be found in [15]. These weak assumptions are assumed to hold for every self-similar process considered in the paper. The exponent  $H$  is called the Hurst parameter and we say  $\{X(t), t \geq 0\}$  is  $H$ -ss or  $H$ -sssi if it also has stationary increments.

Following [13], the definition of multifractal process that we present first is motivated by generalizing the scaling rule of self-similar processes in the following manner:

**Definition 1.** A stochastic process  $\{X(t)\}$  is said to be multifractal if

$$\{X(ct)\} \stackrel{d}{=} \{M(c)X(t)\}, \quad (3)$$

where for every  $c > 0$ ,  $M(c)$  is a random variable independent of  $\{X(t)\}$  whose distribution does not depend on  $t$ .

When  $M(c)$  is deterministic for every  $c > 0$ , the process is self-similar and  $M(c) = c^H$  if the process is nontrivial and stochastically continuous at 0. The scaling factor  $M(c)$  is assumed to satisfy the following property:

$$M(ab) \stackrel{d}{=} M_1(a)M_2(b), \quad (4)$$

for every choice of  $a$  and  $b$ , where  $M_1$  and  $M_2$  are independent copies of  $M$ . This generalizes the property of the deterministic factor for  $H$ -ss processes  $(ab)^H = a^H b^H$ . A motivation for this property can be found in [13].

However, instead of Definition 1, scaling is usually specified in terms of moments. The idea of extracting the scaling properties from average type quantities, like  $L^p$  norm, dates back to the work of Frisch and Parisi ([6]).

**Definition 2.** A stochastic process  $\{X(t)\}$  is said to be multifractal if there exist functions  $c(q)$  and  $\tau(q)$  such that

$$E|X(t) - X(s)|^q = c(q)|t - s|^{\tau(q)}, \quad \forall t, s \in \mathcal{T}, \quad \forall q \in \mathcal{Q}, \quad (5)$$

where  $\mathcal{T}$  and  $\mathcal{Q}$  are intervals on the real line with positive length and  $0 \in \mathcal{T}$ .

The function  $\tau(q)$  is called the scaling function. The set  $\mathcal{Q}$  may also include negative reals. The definition can also be based on the moments of the process instead of the moments of the increments, i.e.  $E|X(t)| = c(q)t^{\tau(q)}$ . If the increments are stationary, these definitions coincide. It is clear that if  $\{X(t)\}$  is  $H$ -sssi, then  $\tau(q) = Hq$  where it is defined. One can also show that  $\tau(q)$  must be concave. Strict concavity can hold only over a finite time horizon, otherwise  $\tau(q)$  would be linear. This is not considered to be a problem for practical purposes (see [13] for details). Since the scaling function is linear for self-similar processes, every departure from linearity can be attributed to multifractality.

However, for this reasoning to make sense, one must assume moment scaling to hold as otherwise self-similarity and multifractality are not complementary notions.

The drawback of involving moments in the definition is that they can be infinite. This narrows the applicability of the definition and, as we show later, can hide the information about the singularity spectrum.

It is easy to see that under stationary increments the defining property (3), along with the property (4), implies multifractality Definition 2. Indeed, (4) implies that  $E|M(c)|^q$  must be of the form  $c^{\tau(q)}$  and the claim follows from  $X(t) \stackrel{d}{=} M(t)X(1)$ . One has to assume finiteness of the moments involved in order for the statements like (5) to have sense. Also notice that both definitions imply  $X(0) = 0$  a.s., which will be used through the paper.

There exist many variations of Definition 2 (see e.g. [10, 16, 17]). Some processes obey the definition only for a small range of values  $t$  or for asymptotically small  $t$ . The stationarity of increments may also be imposed. When referring to multifractality we will make clear which definition we mean. However, we exclude self-similar processes from the preceding definitions.

## 2.1 Detecting Multifractality

An important question related to multifractal processes is to confirm the occurrence of multifractal properties in empirical time series. Definition 2, which is a direct consequence of Definition 1 if (4) is assumed, provides a simple criterion for detecting multifractal stochastic processes. To do this, one must first determine that the moment scaling of the form (5) holds. If this is true, then the method can be based on exploiting the fact that the scaling function is linear for self-similar processes where it is defined. Every departure from linearity can therefore be accredited to multifractality. So, the main problem is to check if the moment scaling holds and then estimate the scaling function from the data

and inspect its shape.

Consider a stationary increments process  $X(t)$  defined for  $t \in [0, T]$  and suppose  $X(0) = 0$ . Divide the interval  $[0, T]$  into  $\lfloor T/t \rfloor$  blocks of length  $t$  and define the partition function (sometimes also called empirical structure function):

$$S_q(T, t) = \frac{1}{\lfloor T/t \rfloor} \sum_{i=1}^{\lfloor T/t \rfloor} |X(it) - X((i-1)t)|^q. \quad (6)$$

If  $\{X(t)\}$  is multifractal with stationary increments then  $ES_q(T, t) = E|X(t)|^q = c(q)t^{\tau(q)}$  and

$$\ln ES_q(T, t) = \tau(q) \ln t + \ln c(q). \quad (7)$$

One can also see  $S_q(T, t)$  as the empirical counterpart of the left-hand side of (5). Suppose that the process is sampled at equidistant time points. We can assume these are the time points  $1, \dots, T$  (see [18]). For fixed value of  $q$ , scaling of moments can be confirmed by plotting the points  $(\ln t_i, \ln S_q(T, t_i))$ ,  $i = 1, \dots, N$  for chosen  $0 \leq t_1 < \dots < t_N \leq T$ . If these are approximately linear, we can suspect (7) to hold and consider  $\tau(q)$  as the slope of the simple linear regression of  $\ln S_q(T, t)$  on  $\ln t$ . Using the well known formula for the slope of the linear regression line, the empirical scaling function is defined as

$$\hat{\tau}_{N,T}(q) = \frac{\sum_{i=1}^N \ln t_i \ln S_q(n, t_i) - \frac{1}{N} \sum_{i=1}^N \ln t_i \sum_{j=1}^N \ln S_q(n, t_j)}{\sum_{i=1}^N (\ln t_i)^2 - \frac{1}{N} \left( \sum_{i=1}^N \ln t_i \right)^2}, \quad (8)$$

where  $N$  is the number of time points chosen in the regression. Repeating the procedure for a range of  $q$  values we obtain a plot of the estimated scaling function. If it is nonlinear, we can suspect multifractal scaling of the underlying process. See [19, 20] for more details on this methodology. It was shown in [18] that a large class of processes behaves as the relation (7) holds even though there is no exact moment scaling (5). Moreover, some processes may appear empirically as multifractal even when there is no some exact scaling



property. We discuss this in more details in Sec. 4.

*Remark 1.* Although the definition (8) follows naturally from the moment scaling relation (5), it is not the only one used in the literature. Another typical choice is to estimate the scaling function by using only the smallest time scale available. For example, for the cascade process on the interval  $[0, T]$  the smallest interval is usually of the length  $2^{-j}T$  for some  $j$ . One can then estimate the scaling function at point  $q$  as

$$\frac{\log_2 S_q(T, 2^{-j}T)}{-j}. \quad (9)$$

In this regime, the asymptotic behaviour of the estimator is usually investigated by letting  $j \rightarrow \infty$ . The estimator (8) estimates the scaling function across different time scales and can therefore be regarded as more general than (9).

## 2.2 Spectrum of Singularities

The preceding notions of multifractality involve “global” properties of the process. Alternatively, one can base the definition on the “local” scaling properties, such as roughness of the process sample paths measured by the pointwise Hölder exponents. There are different approaches on how to develop the notion of a multifractal function. First, we say that a function  $f : [0, \infty) \rightarrow \mathbb{R}$  is  $C^\gamma(t_0)$  if there exists constant  $C > 0$  such that for all  $t$  in some neighborhood of  $t_0$

$$|f(t) - f(t_0)| \leq C|t - t_0|^\gamma.$$

One can also define that  $f$  is Hölder continuous at point  $t_0$  if  $|f(t) - P_{t_0}(t)| \leq C|t - t_0|^\gamma$  for some polynomial  $P_{t_0}$  of degree at most  $\lfloor \gamma \rfloor$ . If  $P_{t_0}$  is constant, then  $P_{t_0} \equiv f(t_0)$  and two definitions coincide. In particular, this happens when  $\gamma < 1$ . For other conditions of equivalence and more details see [10]. In what follows we will use the first definition as in

many cases we consider only processes whose sample paths are  $C^\gamma(t_0)$  with  $\gamma < 1$  at any point  $t_0$ .

A pointwise Hölder exponent of the function  $f$  at  $t_0$  is then

$$H(t_0) = \sup \{ \gamma : f \in C^\gamma(t_0) \}. \quad (10)$$

Consider sets  $S_h = \{t : H(t) = h\}$  where  $f$  has the Hölder exponent of value  $h$ . These sets are usually fractal in the sense that they have non-integer Hausdorff dimension. Define  $d(h)$  to be the Hausdorff dimension of  $S_h$ , using the convention that the dimension of an empty set is  $-\infty$ . The function  $d(h)$  is called the spectrum of singularities (also multifractal or Hausdorff spectrum). We will refer to set of  $h$  such that  $d(h) \neq -\infty$  as the support of the spectrum. A function  $f$  is said to be multifractal if the support of its spectrum is nontrivial, in the sense that it is not a one point set. This is naturally extended to stochastic processes:

**Definition 3.** A stochastic process  $\{X(t)\}$  on some probability space  $(\Omega, \mathcal{F}, P)$  is said to have multifractal paths if for (almost) every  $\omega \in \Omega$ ,  $t \mapsto X(t, \omega)$  is a multifractal function.

When considered for a stochastic process, Hölder exponents are random variables and  $S_h$  random sets. However, in many cases the spectrum is deterministic ([21]). Moreover, the spectrum is usually homogeneous, in the sense it is the same when considered over any nonempty subset  $A \subset [0, \infty)$ . All the examples considered in the following will have these two properties. An example of a process with random, nonhomogeneous spectrum can be found in [22].

## 2.3 Multifractal Formalism

The multifractal formalism relates local and global scaling properties by connecting singularity spectrum with the scaling function via the Legendre transform:

$$d(h) = \inf_q (hq - \tau(q) + 1). \quad (11)$$

Since the Legendre transform is concave, the spectrum is always a concave function, provided the multifractal formalism holds. If the multifractal formalism holds, then  $\inf_q (hq - \tau(q) + 1) = -\infty$  implies that  $S_h = \emptyset$  so that  $h$  is not the Hölder exponent at any point. In addition, the formalism gives the possibility of estimating the spectrum as the Legendre transform of the estimated scaling function.

A substantial work has been done to investigate when this formalism holds. The validity of the formalism depends on which definition of  $\tau$  one uses. Since it ensures that the spectrum can be estimated from computable global quantities, it is a desirable property of the object considered. This is the reason many authors seek for different definitions of global and local scaling properties that would always be related by a certain type of multifractal formalism.

The validity of the multifractal formalism is known to be limited when the scaling function is based on the process increments ([3]). It has been showed that a large class of processes can empirically produce nonlinear scaling function and that this behaviour is influenced by the tail index ([18]). These nonlinearities are not connected with the spectrum, except in the models that posses some scaling property. In many examples negative order moments can also produce concavity in the estimated scaling function since in many models they are infinite. As we will show on the example of self-similar stationary increments processes, divergence of the negative order moments is not related to the spectrum in general. Thus the estimated nonlinearity may be an artefact of the estimation method. We propose a simple modification of the partition function that will

make it more robust. On the other hand, nonlinearity that comes from diverging positive order moments is crucial in estimating the spectrum with (11). For self-similar processes, increments based partition function can capture these nonlinearities correctly.

The wavelets have proved to be a powerful tool in studying multifractality. Instead of using moments, one can base the definition of the scaling function on the wavelet decomposition of the process (see e.g. [10, 23]). This has a direct empirical counterpart based on the estimation of the wavelet coefficients and leads to different methods for multifractal analysis. However, this approach is also sensitive to diverging moments as has been noted in [24] where the wavelet based estimator of the tail index is proposed. The scaling based on the wavelet coefficients is also unable to yield a full spectrum of singularities. In [25], the formalism based on wavelet leaders has been proposed. This in some sense resembles the method we propose in Sec. 5, although our motivation comes from the results given in the next section.

On the other hand, one can also replace the definition of the spectrum to achieve multifractal formalism. For other definitions of the local scaling, such as the one based on the so-called coarse Hölder exponents, see e.g. [10, 26].

The choice of the range over which the infimum in (11) is taken can also be a subject of discussion. From the statistical point of view, moments of negative order are not usually investigated. Sometimes  $\tau(q)$  is calculated only for  $q > 0$  and can therefore yield only left (increasing) part of the spectrum. For more details see [9, 10].

### 3 BOUNDS ON THE SUPPORT OF THE SPECTRUM

The fractional Brownian motion (FBM) is a Gaussian process  $\{B_H(t), t \geq 0\}$ , which starts at zero, has zero expectation for every  $t$  and the following covariance function

$$EB_H(t)B_H(s) = \frac{1}{2} (|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \quad H \in (0, 1).$$

If  $H = 1/2$ , then FBM reduces to the standard Brownian motion (BM). The FBM is  $H$ -sssi and has a trivial spectrum consisting of only one point, i.e.  $d(H) = 1$  and  $d(h) = -\infty$  for  $h \neq H$ . We say that the paths of FBM are monofractal. However, some self-similar processes have nontrivial spectrum. Our goal in this section is to identify the property of the process that makes the spectrum nontrivial. We do this by deriving the bounds on the support of the spectrum. The lower bound is a consequence of the well-known Kolmogorov's continuity theorem. Such applications of Kolmogorov's theorem have appeared in the multifractal literature before (see e.g. [27, Corrolary 5]). For the upper bound we prove a sort of complement of this theorem.

Before we proceed, we fix the following notation for a process  $\{X(t), t \in \mathcal{T}\}$  where  $\mathcal{T} = [0, T]$  or  $\mathcal{T} = [0, +\infty)$ . We denote the range of finite moments as  $\mathcal{Q} = (\underline{q}, \bar{q})$ , i.e.

$$\begin{aligned} \bar{q} &= \sup\{q > 0 : E|X(t)|^q < \infty, \forall t\}, \\ \underline{q} &= \inf\{q < 0 : E|X(t)|^q < \infty, \forall t\}. \end{aligned} \tag{12}$$

If  $\{X(t)\}$  is multifractal in the sense of Definition 2 with the scaling function  $\tau$ , then define

$$\begin{aligned} H^- &= \sup \left\{ \frac{\tau(q)}{q} - \frac{1}{q} : q \in (0, \bar{q}) \text{ \& } \tau(q) > 1 \right\}, \\ H^+ &= \inf \left\{ \frac{\tau(q)}{q} - \frac{1}{q} : q \in (\underline{q}, 0) \text{ \& } \tau(q) < 1 \right\}, \end{aligned} \tag{13}$$

with the convention that  $\sup \emptyset = 0$  and  $\inf \emptyset = +\infty$ . In this context, we always assume that (5) holds on the whole  $\mathcal{T}$  and  $\mathcal{Q}$ . Every process  $\{X(t), t \in \mathcal{T}\}$  considered here is defined on some probability space  $(\Omega, \mathcal{F}, P)$  and measurable, meaning that  $(t, \omega) \mapsto X(t, \omega)$  is  $\mathcal{B}(\mathcal{T}) \times \mathcal{F}$ -measurable. Furthermore, we assume that  $\{X(t), t \in \mathcal{T}\}$  is separable with respect to any dense countable set  $\mathbb{T} \subset \mathcal{T}$ , in the sense that for all  $t \in \mathcal{T}$  there exists a sequence  $(t_n)$  in  $\mathbb{T}$ ,  $t_n \rightarrow t$  such that a.s.  $X(t_n) \rightarrow X(t)$ . We say that the two processes  $\{X(t), t \in \mathcal{T}\}$  and  $\{\tilde{X}(t), t \in \mathcal{T}\}$  defined on the same probability space are modifications of each other if for every  $t \in \mathcal{T}$ ,  $P(X(t) = \tilde{X}(t)) = 1$ . If  $P(X(t) = \tilde{X}(t), \forall t \in \mathcal{T}) = 1$ , then we say that the two processes are indistinguishable. Every stochastic process  $\{X(t), t \in \mathcal{T}\}$  has a separable modification (see e.g. [28]).

### 3.1 The Lower Bound

Using the well-known Kolmogorov's criterion it is easy to derive the lower bound on the support of the spectrum. Before stating the theorem, we define  $f : \mathcal{T} \rightarrow \mathbb{R}$  to be locally Hölder continuous of order  $\gamma$  if for every compact  $K \subset \mathcal{T}$  there exists a constant  $C(K)$  such that

$$|f(t) - f(s)| \leq C(K)|t - s|^\gamma, \quad \forall t, s \in K.$$

It is clear that the local Hölder continuity at some domain implies pointwise Hölder continuity of the same order at any point. The proof of the following theorem can be found in [29, Theorem 2.8] or [30, Theorem 3.23].

**Theorem 1** (Kolmogorov-Chentsov). *Suppose that the process  $\{X(t), t \in \mathcal{T}\}$  satisfies*

$$E|X(t) - X(s)|^\alpha \leq C|t - s|^{1+\beta}, \quad \forall t, s \in \mathcal{T}, \tag{14}$$

*for some constants  $\alpha > 0$ ,  $\beta > 0$  and  $C > 0$ . Then there exists a modification  $\{\tilde{X}(t), t \in \mathcal{T}\}$  of  $\{X(t), t \in \mathcal{T}\}$  having continuous sample paths. Furthermore, a.s.  $\{\tilde{X}(t)\}$  is locally*

Hölder continuous of order  $\gamma$  for every  $\gamma \in (0, \beta/\alpha)$ .

**Proposition 1.** *Suppose  $\{X(t), t \in \mathcal{T}\}$  is multifractal in the sense of Definition 2. If  $\tau(q) > 1$  for some  $q \in (0, \bar{q})$ , then there exists a modification of  $\{X(t)\}$  which is a.s. locally Hölder continuous of order  $\gamma$  for every*

$$\gamma \in \left(0, \frac{\tau(q)}{q} - \frac{1}{q}\right).$$

*In particular, there exists a modification such that a.s.*

$$H^- \leq H(t), \quad \forall t \in \mathcal{T},$$

where  $H(t)$  is defined by (10) and  $H^-$  by (13).

*Proof.* This is a simple consequence of Theorem 1 since Definition 2 implies

$$E|X(t) - X(s)|^q = c(q)|t - s|^{1+(\tau(q)-1)}.$$

For the second part, if  $H^- = 0$  there is nothing to prove. Otherwise, by (13), for each  $\gamma < H^-$  there is  $q \in (0, \bar{q})$  such that  $\tau(q) > 1$  and  $\gamma < (\tau(q) - 1)/q$ , and thus, by the first part there is modification which is a.s. locally Hölder continuous of order  $\gamma$ . Since all continuous modifications are indistinguishable (see e.g. [29, Problem 1.5]), we have the desired modification. This implies that a.s. the pointwise Hölder exponent is everywhere greater than  $H^-$ .  $\square$

In the sequel we always suppose to work with the modification from Proposition 1 where applicable. If  $H^- > 0$ , we conclude that the spectrum  $d(h) = -\infty$  for  $h \in (0, H^-)$ . This way we can establish an estimate for the left endpoint of the support of the spectrum. It also follows that if the process is  $H$ -sssi and has finite moments of every positive order, then  $H^- = H \leq H(t)$ . Thus, when the moment scaling holds, path irregularities are

closely related with infinite moments of positive order. We make this point stronger later.

Theorem 1 is valid for general stochastic processes. Although moment condition (14) is appealing, the condition needed for the proof of Theorem 1 can be stated in a different form.

**Corollary 1.** *For the process  $\{X(t), t \in \mathcal{T}\}$  there exists a modification which is a.s. locally Hölder continuous of order  $\gamma > 0$  if for some  $\eta > 1$  it holds that for every  $K > 0$  there exists  $C > 0$  such that*

$$\limsup_{t \rightarrow 0} \frac{P(|X(s+t) - X(s)| \geq Kt^\gamma)}{t^\eta} \leq C, \quad \forall s \in \mathcal{T}.$$

*Proof.* This is obvious from the proof of Theorem 1; see [29, Theorem 2.8].  $\square$

## 3.2 The Upper Bound

The negative order moments are considered responsible for the right part of the spectrum. We show that this is only partially true, as this depends on whether the negative order moments are finite. To establish the bound on the right endpoint of the support of the spectrum, one needs to show that a.s. the sample paths are nowhere Hölder continuous of some order  $\gamma$ , i.e. that a.s.  $t \mapsto X(t) \notin C^\gamma(t_0)$  for each  $t_0 \in \mathcal{T}$ . To show this we first use a criterion based on the negative order moments, similar to (14). The resulting theorem can be seen as a sort of a complement of the Kolmogorov-Chentsov theorem. The method of proof is similar with the proof of nowhere differentiability of BM (see e.g. [31]). We then apply this result to moment scaling multifractals to get an estimate for the support of the spectrum.

In proving the statements involving negative order moments we use the following two simple facts at several places. The first is a Markov's inequality for negative order



moments. If  $X$  is a random variable,  $\varepsilon > 0$  and  $q < 0$ , then

$$P(|X| \leq \varepsilon) = P(|X|^q \geq \varepsilon^q) \leq \frac{E|X|^q}{\varepsilon^q}.$$

The second fact is the expression for the  $q$ -th order moment,  $q < 0$ ,

$$E|X|^q = - \int_0^\infty qy^{-q-1} P(1/|X| \geq y) dy = - \int_0^\infty qy^{q-1} P(|X| \leq y) dy. \quad (15)$$

**Theorem 2.** *Suppose that the process  $\{X(t), t \in \mathcal{T}\}$  satisfies*

$$E|X(t) - X(s)|^\alpha \leq C|t - s|^{1+\beta}, \quad \forall t, s \in \mathcal{T}, \quad (16)$$

*for some constants  $\alpha < 0$ ,  $\beta < 0$  and  $C > 0$ . Then a.s.  $\{X(t)\}$  is nowhere Hölder continuous of order  $\gamma$  for every  $\gamma > \beta/\alpha$ .*

*Proof.* First, it suffices to prove the statement by fixing arbitrary  $\gamma > \beta/\alpha$ . Indeed, this would give events  $\Omega_\gamma$ ,  $P(\Omega_\gamma) = 0$  such that for  $\omega \in \Omega \setminus \Omega_\gamma$ ,  $t \mapsto X(t, \omega)$  is nowhere Hölder continuous of order  $\gamma$ . If  $\Omega_0$  is the union of  $\Omega_\gamma$  over all  $\gamma \in (\beta/\alpha, \infty) \cap \mathbb{Q}$ , then  $\Omega_0 \in \mathcal{F}$ ,  $P(\Omega_0) = 0$  and  $\Omega \setminus \Omega_0$  would fit the statement of the theorem.

Secondly, it is enough to consider only restrictions to the interval  $[0, 1)$ , as, if needed, for  $n \in \mathbb{N}$  we get from this the proof for the interval  $[n, n+1)$  by using the process  $X'(t) = X(n+t) - X(n)$ ,  $t \in [0, 1)$ . Removing null sets for all  $n \in \mathbb{N}$  would imply the general statement.

For  $j, k \in \mathbb{N}$  define the set

$$M_{jk} := \bigcup_{t \in [0, 1)} \bigcap_{h \in [0, 1/k]} \{\omega \in \Omega : |X(t+h, \omega) - X(t, \omega)| \leq jh^\gamma\}.$$

It is clear that if  $\omega \notin M_{jk}$  for every  $j, k \in \mathbb{N}$ , then  $t \mapsto X(t, \omega)$  is nowhere Hölder continuous of order  $\gamma$ . As there are countably many  $M_{jk}$ , it is enough to fix arbitrary

$j, k \in \mathbb{N}$  and show that  $M_{jk} \subset A$  for some  $A \in \mathcal{F}$  such that  $P(A) = 0$ .

Suppose  $n > 2k$  and  $\omega \in M_{jk}$ . Then there is some  $t \in [0, 1)$  such that

$$|X(t+h, \omega) - X(t, \omega)| \leq jh^\gamma, \quad \forall h \in [0, 1/k]. \quad (17)$$

Take  $i \in \{1, \dots, n\}$  such that

$$\frac{i-1}{n} \leq t < \frac{i}{n}. \quad (18)$$

Since  $n > 2k$  we have

$$0 \leq \frac{i}{n} - t < \frac{i+1}{n} - t \leq \frac{i+1}{n} - \frac{i-1}{n} = \frac{2}{n} < \frac{1}{k},$$

and from (17) it follows that

$$\begin{aligned} \left| X\left(\frac{i+1}{n}, \omega\right) - X\left(\frac{i}{n}, \omega\right) \right| &\leq \left| X\left(\frac{i+1}{n}, \omega\right) - X(t, \omega) \right| + \left| X(t, \omega) - X\left(\frac{i}{n}, \omega\right) \right| \\ &\leq 2^{\gamma+1} j n^{-\gamma}. \end{aligned}$$

Put  $A_i^{(n)} = \{|X(\frac{i+1}{n}) - X(\frac{i}{n})| \leq 2^{\gamma+1} j n^{-\gamma}\}$ . Since  $\omega$  was arbitrary it follows that

$$M_{jk} \subset \bigcup_{i=1}^n A_i^{(n)}.$$

Using Markov's inequality for  $\alpha < 0$  and the assumption of the theorem we get

$$\begin{aligned} P(A_i^{(n)}) &\leq \frac{E|X(\frac{i+1}{n}) - X(\frac{i}{n})|^\alpha}{(2^{\gamma+1} j)^\alpha n^{-\gamma\alpha}} \leq C(2^{\gamma+1} j)^{-\alpha} n^{\gamma\alpha-1-\beta}, \\ P\left(\bigcup_{i=1}^n A_i^{(n)}\right) &\leq \sum_{i=1}^n P(A_i^{(n)}) \leq C(2^{\gamma+1} j)^{-\alpha} n^{-(\beta-\gamma\alpha)}. \end{aligned} \quad (19)$$

If we set

$$A = \bigcap_{n>2k} \bigcup_{i=1}^n A_i^{(n)},$$

then  $A \in \mathcal{F}$  and  $M_{jk} \subset A$ . Since  $\gamma > \beta/\alpha$ , it follows that  $\beta - \gamma\alpha > 0$  and hence  $P(A) = 0$ .

This proves the theorem.  $\square$

**Proposition 2.** *Suppose  $\{X(t), t \in \mathcal{T}\}$  is multifractal in the sense of Definition 2. If  $\tau(q) < 1$  for some  $q \in (\underline{q}, 0)$ , then a.s.  $\{X(t)\}$  is nowhere Hölder continuous of order  $\gamma$  for every*

$$\gamma \in \left( \frac{\tau(q)}{q} - \frac{1}{q}, +\infty \right).$$

*In particular, a.s.*

$$H(t) \leq H^+, \quad \forall t \in \mathcal{T}.$$

*Proof.* Definition 2 implies

$$E|X(t) - X(s)|^q = c(q)|t - s|^{1+(\tau(q)-1)}.$$

Since  $q < 0$ ,  $\tau(q) - 1 < 0$  and the statement follows from Theorem 2.  $\square$

This proposition shows that  $d(h) = -\infty$  for  $h \in (H^+, \infty)$ . Recall that  $H^+$  is defined in (13).

*Remark 2.* Statements like the ones in Proposition 1 and 2 are stronger than saying, for example, that for every  $t \in \mathcal{T}$ ,  $H(t) \leq U$  a.s. Indeed, an application of the Fubini's theorem would yield that for almost every path,  $H(t) \leq U$  for almost every  $t$ . If we put  $h = U + \delta$ , then the Lebesgue measure of the set  $S_h = \{t : H(t) = h\}$  is zero a.s. This, however, does not imply that  $d(h) = -\infty$  and hence, it is impossible to say something about the spectrum of almost every sample path. On the other hand, it is clear that this type of statements are implied by Propositions 1 and 2.

For an example of this weaker type of the bound, consider  $\{X(t), t \in \mathcal{T}\}$  multifractal in the sense of Definition 2. If there is  $q \in (\underline{q}, 0)$ , then for every  $t \in \mathcal{T}$

$$H(t) \leq \frac{\tau(q)}{q} \text{ a.s.}$$

Indeed, let  $\delta > 0$  and suppose  $C > 0$ . Since  $q < 0$ , by Markov's inequality

$$P\left(|X(t + \varepsilon) - X(t)| \leq C|\varepsilon|^{\frac{\tau(q)}{q} + \delta}\right) \leq \frac{E|X(t + \varepsilon) - X(t)|^q}{C^q|\varepsilon|^{\tau(q) + \delta q}} = \frac{c(q)}{C^q|\varepsilon|^{\delta q}} \rightarrow 0,$$

as  $\varepsilon \rightarrow 0$ . We can choose a sequence  $(\varepsilon_n)$  that converges to zero such that

$$P\left(|X(t + \varepsilon_n) - X(t)| \leq C|\varepsilon_n|^{\frac{\tau(q)}{q} + \delta}\right) \leq \frac{1}{2^n}.$$

Now, by the Borel-Cantelli lemma

$$\frac{|X(t + \varepsilon_n) - X(t)|}{|\varepsilon_n|^{\frac{\tau(q)}{q} + \delta}} \rightarrow \infty \text{ a.s., as } n \rightarrow \infty.$$

Thus, for arbitrary  $\delta > 0$  it holds that for every  $t$ ,  $H(t) \leq \frac{\tau(q)}{q} + \delta$  a.s. However, this result does not allow us to say anything about the spectrum.

Consider for the moment the FBM. The range of finite moments is  $(-1, \infty)$  and  $\tau(q) = Hq$  for  $q \in (-1, \infty)$ , so we have  $H^+ = H + 1$ . Thus, the best we can say from Proposition 2, is that  $d(h) = -\infty$  for  $h > H + 1$ . However, we know that  $d(h) = -\infty$  for  $h > H$ . If the infimum in the definition of  $H^+$  could be considered over all negative  $q$ , we would get exactly the right endpoint of the support of the spectrum.

The fact that the bound derived in Proposition 2 is not sharp enough for some examples points that negative order moments may not be the right paradigm to explain the spectrum. We therefore provide more general conditions that do not depend on the finiteness of moments.

**Theorem 3.** *A process  $\{X(t), t \in \mathcal{T}\}$  is a.s. nowhere Hölder continuous of order  $\gamma > 0$  if for some  $\eta > 1$  and  $m \in \mathbb{N}$  it holds that for every  $K > 0$  there exists  $C > 0$  such that*

$$\limsup_{t \rightarrow 0} \frac{P\left(\max_{l=1, \dots, m} |X(s + lt) - X(s + (l-1)t)| \leq Kt^\gamma\right)}{t^\eta} \leq C, \quad \forall s \in \mathcal{T}. \quad (20)$$

*Proof.* The first part of the proof goes exactly as in the proof of Theorem 2. Fix  $j, k \in \mathbb{N}$  and take  $n \in \mathbb{N}$  such that

$$n > (m+1)k.$$

If  $\omega \in M_{jk}$ , then there is some  $t \in [0, 1)$  and  $i \in \{1, \dots, n\}$  such that (17) and (18) hold.

Choice of  $n$  ensures that for  $l \in \{1, \dots, m\}$

$$0 < \frac{i+l-1}{n} - t < \frac{i+l}{n} - t < \frac{i+l}{n} - \frac{i-1}{n} = \frac{l+1}{n} \leq \frac{1}{k}.$$

It follows from (17) that for each  $l \in \{1, \dots, m\}$

$$\left| X\left(\frac{i+l}{n}, \omega\right) - X\left(\frac{i+l-1}{n}, \omega\right) \right| \leq j \left(\frac{l+1}{n}\right)^\gamma + j \left(\frac{l}{n}\right)^\gamma \leq 2j \left(\frac{m+1}{n}\right)^\gamma.$$

Let

$$\begin{aligned} A_{i,l}^{(n)} &= \left\{ \left| X\left(\frac{i+l}{n}\right) - X\left(\frac{i+l-1}{n}\right) \right| \leq 2j \left(\frac{m+1}{n}\right)^\gamma \right\}, \\ A_i^{(n)} &= \bigcap_{l=1}^m A_{i,l}^{(n)}. \end{aligned}$$

It then follows that

$$M_{jk} \subset \bigcup_{i=1}^n A_i^{(n)}.$$

From the assumption, there exists  $C > 0$  such that

$$\begin{aligned} P(A_i^{(n)}) &= P\left(\max_{l=1, \dots, m} \left| X\left(\frac{i+l}{n}\right) - X\left(\frac{i+l-1}{n}\right) \right| \leq 2j(m+1)^\gamma \left(\frac{1}{n}\right)^\gamma\right) \leq Cn^{-\eta}, \\ P\left(\bigcup_{i=1}^n A_i^{(n)}\right) &\leq \sum_{i=1}^n P(A_i^{(n)}) \leq Cn^{-(\eta-1)}. \end{aligned}$$

Now setting

$$A = \bigcap_{n > (m+1)k} \bigcup_{i=1}^n A_i^{(n)} \in \mathcal{F},$$

it follows that  $P(A) = 0$ , since  $\eta > 1$ .  $\square$

The following simple corollary may also be established directly from the proof of Theorem 2, Eq. (19).

**Corollary 2.** *A process  $\{X(t), t \in \mathcal{T}\}$  is a.s. nowhere Hölder continuous of order  $\gamma > 0$  if for some  $\eta > 1$  it holds that for every  $K > 0$  there exists  $C > 0$  such that*

$$\limsup_{t \rightarrow 0} \frac{P(|X(s+t) - X(s)| \leq Kt^\gamma)}{t^\eta} \leq C, \quad \forall s \in \mathcal{T}.$$

Theorem 3 enables one to avoid using moments in deriving the bound. As an example, we consider how Theorem 3 can be applied in the simple case when  $\{X(t)\}$  is BM. Since  $\{X(t)\}$  is 1/2-sssi we have

$$P\left(\max_{l=1,\dots,m} |X(lt) - X((l-1)t)| \leq Kt^\gamma\right) = P\left(\max_{l=1,\dots,m} |X(l) - X(l-1)| \leq Kt^{\gamma-1/2}\right).$$

Due to independent increments:

$$P\left(\max_{l=1,\dots,m} |X(l) - X(l-1)| \leq Kt^{\gamma-1/2}\right) = (P(|X(1)| \leq Kt^{\gamma-1/2}))^m \leq Ct^{m(\gamma-1/2)}.$$

This holds for every  $\gamma > 1/2$  and  $m \in \mathbb{N}$  and by taking  $m > 1/(\gamma - 1/2)$  we conclude that  $d(h) = -\infty$  for  $h > 1/2$ .

Before we proceed on applying these results, we state the following simple corollary that expresses the criterion (20) in terms of the negative order moments, but now moments of the maximum of increments. This is a generalization of Theorem 2, which enables bypassing infinite negative order moments under very general conditions. In the next section we apply this criterion to  $H$ -sssi processes.

**Corollary 3.** *Suppose that a process  $\{X(t), t \in \mathcal{T}\}$  satisfies*

$$E \left[ \max_{l=1, \dots, m} |X(s+lt) - X(s+(l-1)t)| \right]^\alpha \leq Ct^{1+\beta}, \quad \forall t, s \in \mathcal{T}, \quad (21)$$

*for some  $\alpha < 0$ ,  $\beta < 0$ ,  $m \in \mathbb{N}$  and  $C > 0$ . Then a.s.  $\{X(t)\}$  is nowhere Hölder continuous of order  $\gamma$  for every  $\gamma > \beta/\alpha$ .*

*Proof.* This follows directly from the Markov's inequality for negative order moments and Theorem 3 since

$$\begin{aligned} P \left( \max_{l=1, \dots, m} |X(s+lt) - X(s+(l-1)t)| \leq Kt^\gamma \right) \\ \leq K^{-\alpha} t^{-\gamma\alpha} E \left[ \max_{l=1, \dots, m} |X(s+lt) - X(s+(l-1)t)| \right]^\alpha \leq K^{-\alpha} Ct^{-\alpha\gamma+1+\beta}, \end{aligned}$$

and  $1 + \beta - \alpha\gamma > 1$ . □

### 3.3 The Case of Self-similar Stationary Increments Processes

In this subsection we refine our results for the case of  $H$ -sssi processes by using Corollary 3. These results can also be viewed in the light of the classical papers [32] and [33]. To be able to apply Corollary 3, we need to make sure that the moment in (21) can be made finite by choosing  $m$  large enough. We state this condition explicitly for reference.

**Condition 1.** *Suppose  $\{X(t), t \in \mathcal{T}\}$  is a stationary increments process. For every  $\alpha < 0$  there is  $m_0 \in \mathbb{N}$  such that*

$$E \left[ \max_{l=1, \dots, m_0} |X(l) - X(l-1)| \right]^\alpha < \infty.$$

Unfortunately, Condition 1 is not always easy to check on the specific examples. However, if the process has independent increments, then the following criterion may be useful. It applies for example to Brownian motion and stable Lévy motion.

**Lemma 1.** *If  $\{X(t), t \in \mathcal{T}\}$  has stationary independent increments and  $P(|X(1) - X(0)| \leq t) = O(t^r)$  for some  $r > 0$  as  $t \rightarrow 0$ , then Condition 1 holds.*

*Proof.* From (15) we have that

$$\begin{aligned} E \left[ \max_{l=1, \dots, m} |X(l) - X(l-1)| \right]^\alpha &= - \int_0^\infty \alpha y^{\alpha-1} P \left( \max_{l=1, \dots, m} |X(l) - X(l-1)| \leq y \right) dy \\ &= - \int_0^\infty \alpha y^{\alpha-1} P(|X(l) - X(l-1)| \leq y)^m dy \\ &\leq - \int_0^\varepsilon \alpha y^{\alpha-1} C^m y^{mr} dy + \int_\varepsilon^\infty \alpha y^{\alpha-1} dy < \infty, \end{aligned}$$

by taking  $m$  large enough. □

*Remark 3.* Two examples may provide a deeper insight into Condition 1, as in these examples Condition 1 fails to hold. First, if  $X(t) = tX$  for some random variable  $X$ , then

$$\max_{l=1, \dots, m} |X(l) - X(l-1)| = X$$

and thus, Condition 1 depends on the range of finite moments of  $X$ . For the second example, suppose  $X(t) = \sum_{i=1}^{\lfloor t \rfloor} \xi_i$ , where  $\xi_i$ ,  $i \in \mathbb{N}$ , is an i.i.d. sequence such that  $P(|\xi_1| \leq x) = -\ln 2 / \ln x$  for  $x \in (0, 1/2)$ . This implies, in particular, that  $E|\xi_1|^r = \infty$  for any  $r < 0$ . Moreover,

$$\begin{aligned} E \left[ \max_{l=1, \dots, m} |X(l) - X(l-1)| \right]^\alpha &= - \int_0^\infty \alpha y^{\alpha-1} P \left( \max_{l=1, \dots, m} |X(l) - X(l-1)| \leq y \right) dy \\ &= - \int_0^{1/2} \alpha y^{\alpha-1} \frac{(\ln 2)^m}{(\ln y)^m} dy + \int_{1/2}^\infty \alpha y^{\alpha-1} dy = \infty, \end{aligned}$$

for every  $\alpha < 0$  and  $m \in \mathbb{N}$ , thus Condition 1 does not hold.

We next prove a general theorem about  $H$ -sssi processes.

**Theorem 4.** *Suppose  $\{X(t), t \geq 0\}$  is  $H$ -sssi stochastic process such that Condition 1*



holds and  $H - 1/\bar{q} \geq 0$ . Then a.s.

$$H - \frac{1}{\bar{q}} \leq H(t) \leq H, \quad \forall t \geq 0.$$

*Proof.* By the same argument as in the beginning of the proof of Theorem 2 it is enough to take arbitrary  $\gamma > H$ . Given  $\gamma$  we take  $\alpha < 1/(H - \gamma) < 0$  which implies  $\gamma > H - 1/\alpha$ . Due to Condition 1, we can choose  $m_0 \in \mathbb{N}$  such that  $E[\max_{l=1, \dots, m_0} |X(lt) - X((l-1)t)|]^\alpha < \infty$ . Self-similarity then implies that

$$E\left[\max_{l=1, \dots, m_0} |X(lt) - X((l-1)t)|\right]^\alpha = t^{H\alpha} E\left[\max_{l=1, \dots, m_0} |X(l) - X(l-1)|\right]^\alpha = Ct^{1+(H\alpha-1)}.$$

The claim now follows immediately from Corollary 3 with  $\beta = H\alpha - 1$  since  $\gamma > \beta/\alpha$ .  $\square$

A simple consequence of the preceding is the following statement.

**Corollary 4.** *Suppose that Condition 1 holds. A  $H$ -sssi process with all positive order moments finite has a trivial spectrum, i.e.  $d(h) = -\infty$  for  $h \neq H$ .*

*Remark 4.* From Corollary 4 we conclude that, under very general conditions, a self-similar stationary increments process with a nontrivial spectrum must be heavy-tailed. This shows clearly how infinite moments can affect path properties when the scaling property holds.

The following simple result shows how the nontrivial spectrum of a self-similar stationary increments process implies infinite moments of positive order.

**Proposition 3.** *Suppose  $\{X(t), t \geq 0\}$  is  $H$ -sssi. If  $\gamma < H$  and  $d(\gamma) \neq -\infty$ , then  $E|X(1)|^q = \infty$  for  $q > 1/(H - \gamma)$ .*

*Proof.* Suppose  $E|X(t)|^q < \infty$  for  $q > 1/(H - \gamma)$ . Then for  $\varepsilon > 0$  we can apply Markov's

inequality to get

$$P(|X(t)| \geq Kt^\gamma) = P(|X(1)| \geq Kt^{\gamma-H}) \leq \frac{E|X(1)|^{\frac{1}{H-\gamma}+\varepsilon}}{K^{\frac{1}{H-\gamma}+\varepsilon}t^{-1-\varepsilon(H-\gamma)}} \leq Ct^{1+\varepsilon(H-\gamma)}.$$

By Corollary 1 this implies  $d(\gamma) = -\infty$ , which is a contradiction.  $\square$

### 3.4 The Case of Multifractal Processes

Our next goal is to show that in the definition (13) of  $H^+$  one can essentially take the infimum over all  $q < 0$ . At the moment this makes no sense as  $\tau$  from Definition 2 may not be defined in this range. It is therefore necessary to redefine the meaning of the scaling function and thus we work with the more general Definition 1.

In the next section we will see on the example of the log-normal cascade process that when the multifractal process has all negative order moments finite, the bound derived in Proposition 2 is sharp. In general, this would not be the case for any multifractal in the sense of Definition 1. Take for example a multifractal random walk (MRW), which is a compound process  $X(t) = B(\theta(t))$  where  $B$  is BM and  $\theta$  is an independent cascade process, say log-normal cascade (see [34]). The multifractality of the cascade for  $t < 1$ ,  $\theta(t) \stackrel{d}{=} M(t)\theta(1)$  and multifractality of MRW imply that  $X(t) \stackrel{d}{=} (M(t)\theta(1))^{1/2}B(1)$ . Now by the independence of  $B$  and  $\theta$ , if  $E|B(1)|^q = \infty$ , then  $E|X(t)|^q = \infty$ . Since  $B(1)$  is Gaussian, the moments will be infinite for  $q \leq -1$ .

We thus provide a more general bound which only has a restriction on the moments of the random factor from Definition 1. Therefore, if the process satisfies Definition 1 and if the random factor  $M$  is multifractal by Definition 2 with scaling function  $\tau$ , we define

$$\widetilde{H^+} = \inf \left\{ \frac{\tau(q)}{q} - \frac{1}{q} : q < 0 \text{ \& } E|M(t)|^q < \infty \right\}.$$

**Corollary 5.** *Suppose  $\{X(t), t \in \mathcal{T}\}$  has stationary increments and Condition 1 holds.*

Suppose also it is multifractal by Definition 1 and the random factor  $M$  satisfies Definition 2 with scaling function  $\tau$ . If  $E|M(t)|^q < \infty$ ,  $\forall t \in \mathcal{T}$  for some  $q < 0$ , then a.s.  $\{X(t)\}$  is nowhere Hölder continuous of order  $\gamma$  for every

$$\gamma \in \left( \frac{\tau(q)}{q} - \frac{1}{q}, +\infty \right).$$

In particular, a.s.

$$H(t) \leq \widetilde{H}^+, \quad \forall t \in \mathcal{T}.$$

*Proof.* By Condition 1, for  $m$  large enough it follows from the multifractal property (3) that

$$E \left[ \max_{l=1, \dots, m} |X(lt) - X((l-1)t)| \right]^q = E|M(t)|^q E \left[ \max_{l=1, \dots, m} |X(l) - X(l-1)| \right]^q = Ct^{1+\tau(q)-1}.$$

The claim now follows from Corollary 3 with  $\alpha = q$  and  $\beta = \tau(q) - 1$  and by the argument at the beginning of the proof of Theorem 2.  $\square$

In summary, we provide bounds on the support of the multifractal spectrum. We show that the lower bound can be derived using positive order moments and link infinite moments with path properties for the case of  $H$ -sssi process. In general, negative order moments are not appropriate for explaining the right part of the spectrum. To derive an upper bound on the support of the spectrum, we use negative order moments of the maximum of increments. This may avoid the nonexistence of the negative order moments, which is a property of the distribution itself.

## 4 EXAMPLES

In this section we list several examples of stochastic processes and investigate different aspects of multifractality listed in Sec. 2. We show how the results of Sec. 3 apply in these

cases and also discuss how the multifractal formalism could be achieved. Definitions and further details on the processes considered are given in the Appendix.

## 4.1 Self-similar Processes

It follows from Theorem 4 and Corollary 4 that if  $H$ -sssi process satisfies Condition 1 and has finite positive order moments, then the spectrum is simply

$$d(h) = \begin{cases} 1, & \text{if } h = H \\ -\infty, & \text{otherwise.} \end{cases}$$

This applies to e.g. BM. We conjecture that the same holds for the class of Hermite processes (see e.g. [35, Sec. 7]), however, we were not able to check Condition 1 in this case. The spectrum of Hermite processes has been studied numerically in ([36]). We now discuss heavy tailed examples of  $H$ -sssi processes.

### 4.1.1 Stable Lévy Motion

Suppose  $\{X(t)\}$  is an  $\alpha$ -stable Lévy motion. This process is  $1/\alpha$ -sssi and moment scaling (5) holds but makes sense only for a range of finite moments, that is for  $\mathcal{Q} = (-1, \alpha)$  in Definition 2. For this range of  $q$ , scaling function is  $\tau_{SLM}(q) = q/\alpha$  and the process is self-similar. Due to infinite moments beyond order  $\alpha$  the empirical scaling function (8) will asymptotically behave for  $q > 0$  as

$$\tau_{\alpha}^{\infty}(q) = \begin{cases} \frac{q}{\alpha}, & \text{if } 0 < q \leq \alpha, \\ 1, & \text{if } q > \alpha. \end{cases}$$

See [18] for the precise result. The nonlinearity points that the process would empirically behave as multifractal. The spectrum of singularities is given by ([14]):

$$d_{SLM}(h) = \begin{cases} \alpha h, & \text{if } h \in [0, 1/\alpha], \\ -\infty, & \text{if } h > 1/\alpha. \end{cases} \quad (22)$$

Hence the spectrum is nontrivial and supported on  $[0, 1/\alpha]$ . These are exactly the bounds given in Theorem 4 as in this case  $H = 1/\alpha$  and  $\bar{q} = \alpha$ . We stress that even self-similar processes can have multifractal paths and that this is closely related with infinite moments.

We now discuss which form of the scaling function would yield the multifractal spectrum via the Legendre transform. This will highly depend on the range of  $q$  over which the infimum in the Legendre transform is taken. If we consider Legendre transforms of  $\tau_{SLM}$  and  $\tau_\alpha^\infty$  and take infimum over all positive  $q$  where they are defined, then one can easily check that

$$\inf_{0 < q < \alpha} (hq - \tau_{SLM}(q) + 1) = \inf_{0 < q < \infty} (hq - \tau_\alpha^\infty(q) + 1) = \begin{cases} \alpha h, & \text{if } h \in [0, 1/\alpha], \\ 1, & \text{if } h > 1/\alpha. \end{cases}$$

This actually coincides with the true spectrum (22), except for the part  $h > 1/\alpha$ , which is the infimum obtained when  $q \rightarrow 0$ . To correctly estimate this part one needs negative order moments, which will be discussed later. So even though the moments beyond order  $\alpha$  are infinite, estimating infinite moments with the partition function can lead to the correct spectrum of singularities.

#### 4.1.2 Linear Fractional Stable Motion

In the same manner we treat linear fractional stable motion (LFSM) (see Appendix for the definition). The dependence introduces a new parameter in the scaling relations and the spectrum. The LFSM  $\{X(t)\}$  is  $H$ -sssi and for the range of finite moments  $\mathfrak{Q} = (-1, \alpha)$  scaling function is  $\tau_{LFSM}(q) = Hq$ . As follows from the results of [37] (see also [38], [39] and [40]), empirical scaling function asymptotically behaves for  $q > -1$  as

$$\tau_{H,\alpha}^\infty(q) = \begin{cases} Hq, & \text{if } 0 < q \leq \alpha, \\ \left(H - \frac{1}{\alpha}\right)q + 1, & \text{if } q > \alpha. \end{cases}$$

The combined influence of infinite moments and dependence produces concavity, pointing to multifractality in the empirical sense. In [21], the spectrum was established for  $\alpha \in [1, 2)$ ,  $H \in (0, 1)$  and the long-range dependence case  $H > 1/\alpha$ :

$$d_{LFSM}(h) = \begin{cases} \alpha(h - H) + 1, & \text{if } h \in [H - \frac{1}{\alpha}, H], \\ -\infty, & \text{otherwise.} \end{cases} \quad (23)$$

It is known that in the case  $H < 1/\alpha$  the sample paths are nowhere bounded, which explains the assumptions. Since  $\bar{q} = \alpha$  is the tail index, Theorem 4 gives sharp bounds on the support of the spectrum provided that the Condition 1 holds.

Considering Legendre transform of  $\tau_{LFSM}$  over  $(0, \alpha)$  gives

$$\inf_{0 < q < \alpha} (hq - \tau_{LFSM}(q) + 1) = \begin{cases} \alpha(h - H) + 1, & \text{if } h \in [0, H], \\ 1, & \text{if } h > H. \end{cases}$$

Although the expression is similar to the true spectrum  $d_{LFSM}$  defined in (23), the support

is different. On the other hand, it is easy to check that

$$\inf_{0 < q < \infty} (hq - \tau_{H,\alpha}^\infty(q) + 1) = \begin{cases} -\infty, & \text{if } h < H - 1/\alpha, \\ \alpha(h - H) + 1, & \text{if } h \in [H - 1/\alpha, H], \\ 1, & \text{if } h > H. \end{cases}$$

Thus, the empirical scaling function will lead to a correct left part of the spectrum using formalism. This reveals that the validity of the formalism may be limited if  $\tau$  is specified as in (5). Secondly, it shows the potential of the empirical scaling function and indicates how infinite positive order moments are related with path properties when some scaling property holds.

#### 4.1.3 Inverse Stable Subordinator

The inverse stable subordinator  $\{X(t)\}$  is a non-decreasing  $\alpha$ -ss stochastic process, for some  $\alpha \in (0, 1)$ . The application of the results of the previous section for the inverse stable subordinator is not straightforward as it has nonstationary increments, yet we can prove that it has a trivial spectrum such that  $d(\alpha) = 1$ .

To derive the lower bound we use Theorem 1. First recall that  $a^\alpha + b^\alpha \leq (a + b)^\alpha$  for  $a, b \geq 0$  and  $\alpha \in (0, 1)$ . Taking  $a = t - s$ ,  $b = s$  when  $t \geq s$  and  $a = t$ ,  $b = s - t$  when  $t < s$  gives that  $|t^\alpha - s^\alpha| \leq |t - s|^\alpha$ . Since  $\{X(t)\}$  has finite moments of every positive order we have for arbitrary  $q > 0$  and  $t, s > 0$

$$E|X(t) - X(s)|^q = |t^\alpha - s^\alpha|^q E|X(1)|^q \leq E|X(1)|^q |t - s|^{1+\alpha q-1}.$$

By Theorem 1 there exists modification which is a.s. locally Hölder continuous of order  $\gamma < \alpha - 1/q$ . Since  $q$  can be taken arbitrarily large, we can get the modification such that a.s.  $H(t) \geq \alpha$  for every  $t \geq 0$ .

For the upper bound we use Theorem 3. Given  $\gamma > \alpha$  we choose  $m \in \mathbb{N}$  such that  $m > 1/(\gamma - \alpha)$ . If  $\{Y(t)\}$  is the corresponding stable subordinator, from the property  $\{X(t) \leq a\} = \{Y(a) \geq t\}$  we have for every  $t_1 < t_2$  and  $a > 0$

$$\{X(t_2) - X(t_1) \leq a\} = \{Y(X(t_1) + a) \geq t_2\} = \{Y(X(t_1) + a) - t_1 \geq t_2 - t_1\}.$$

By [41, Theorem 4, p. 77], for every  $t_1 > 0$ ,  $P(Y(X(t_1)) > t_1) = 1$ , thus, on this event

$$\{Y(X(t_1) + a) - t_1 \geq t_2 - t_1\} \subset \{Y(X(t_1) + a) - Y(X(t_1)) \geq t_2 - t_1\}.$$

Now by the strong Markov property choosing  $t$  small enough and stationarity of increments of  $\{Y(t)\}$  we have

$$\begin{aligned} & P\left(\max_{l=1,\dots,m} |X(s+lt) - X(s+(l-1)t)| \leq Kt^\gamma\right) \\ &= P(X(s+t) - X(s) \leq Kt^\gamma, \dots, X(s+mt) - X(s+(m-1)t) \leq Kt^\gamma) \\ &\leq P(Y(X(s) + Kt^\gamma) - Y(X(s)) \geq t, \dots, Y(X(s+(m-1)t) + Kt^\gamma) - Y(X(s+(m-1)t)) \geq t) \\ &\leq (P(Y(Kt^\gamma) \geq t))^m = \left(P\left(Y(1) \geq K^{-\frac{1}{\alpha}}t^{1-\frac{\gamma}{\alpha}}\right)\right)^m \leq (Ct^{\gamma-\alpha})^m, \end{aligned}$$

by the regular variation of the tail for  $t$  sufficiently small. Due to the choice of  $m$ ,  $m(\gamma - \alpha) > 1$ . This property of the first-passage process has been noted in [41, p. 96].

## 4.2 Lévy Processes

Suppose  $\{X(t), t \geq 0\}$  is a Lévy process. The Lévy processes in general do not satisfy the moment scaling of the form (5). The only such examples are the BM and the  $\alpha$ -stable Lévy motion. However, it was shown in [18] that the data sampled from certain Lévy processes may behave as obeying the scaling relation (7) in the sense that one can form a plausible linear regression model relating  $\ln S_q(T, t)$  and  $\ln t$ . More precisely, if  $X(1)$  is



zero mean with heavy-tailed distribution with tail index  $\alpha$  and if  $t_i$  in (8) is of the form  $T^{\frac{i}{N}}$  for  $i = 1, \dots, N$ , then for every  $q > 0$  as  $T, N \rightarrow \infty$  the empirical scaling function will asymptotically behave as

$$\tau_{LP}^\infty(q) = \begin{cases} \frac{q}{\alpha}, & \text{if } 0 < q \leq \alpha \text{ \& } \alpha \leq 2, \\ 1, & \text{if } q > \alpha \text{ \& } \alpha \leq 2, \\ \frac{q}{2}, & \text{if } 0 < q \leq \alpha \text{ \& } \alpha > 2, \\ \frac{q}{2} + \frac{2(\alpha-q)^2(2\alpha+4q-3\alpha q)}{\alpha^3(2-q)^2}, & \text{if } q > \alpha \text{ \& } \alpha > 2. \end{cases} \quad (24)$$

See [18] and [42] for the proof and more details. This shows that estimating the scaling function under infinite moments is influenced by the value of the tail index  $\alpha$  and will yield a concave shape of the scaling function.

The local regularity of Lévy processes has been established in [14] and extended in [21] under weaker assumptions. Denote by  $\beta$  the Blumenthal-Gettoor (BG) index of a Lévy process, i.e.

$$\beta = \inf \left\{ \gamma \geq 0 : \int_{|x| \leq 1} |x|^\gamma \pi(dx) < \infty \right\},$$

where  $\pi$  is the corresponding Lévy measure. If  $\sigma$  is a Brownian component of the characteristic triplet, define

$$\beta' = \begin{cases} \beta, & \text{if } \sigma = 0, \\ 2, & \text{if } \sigma \neq 0. \end{cases}$$

The multifractal spectrum of the Lévy process is given by

$$d_{LP}(h) = \begin{cases} \beta h, & \text{if } h \in [0, 1/\beta'), \\ 1, & \text{if } h = 1/\beta', \\ -\infty, & \text{if } h > 1/\beta'. \end{cases} \quad (25)$$

Thus the most Lévy processes have a nontrivial spectrum. Moreover, the estimated scaling function and the spectrum are not related as they depend on the different parts of the Lévy measure. The behaviour of the estimated scaling function is governed by the tail index which depends on the behaviour of the Lévy measure at infinity since for  $q > 0$ ,  $E|X(1)|^q < \infty$  is equivalent to  $\int_{|x|>1} |x|^q \pi(dx) < \infty$ . On the other hand, the spectrum is determined by the behaviour of  $\pi$  around origin, i.e. by the BG index. The discrepancy happens as there is no exact scaling in the sense of (3) or (5). It is therefore important to check the validity of relation (7) from the data. This may be problematic as it is hard to distinguish exact scaling from the asymptotic one exhibited by a large class of processes.

As there is no exact moment scaling, Propositions 1 and 2 generally do not hold. Thus, in order to establish bounds on the support of the spectrum we use other criteria from Sec. 3. We present two analytically tractable examples to illustrate the use of these criteria.

#### 4.2.1 Inverse Gaussian Lévy Process

The inverse Gaussian Lévy process is a subordinator such that  $X(1)$  has an inverse Gaussian distribution  $IG(\delta, \lambda)$ ,  $\delta > 0, \lambda \geq 0$ , given by the density

$$f(x) = \frac{\delta}{\sqrt{2\pi}} e^{\delta\lambda} x^{-3/2} \exp \left\{ -\frac{1}{2} \left( \frac{\delta^2}{x} + \lambda^2 x \right) \right\}, \quad x > 0.$$

The expression for the cumulant function reveals that for each  $t$ ,  $X(t)$  has  $IG(t\delta, \lambda)$  distribution. The Lévy measure is absolutely continuous with the density given by

$$g(x) = \frac{\delta}{\sqrt{2\pi}} x^{-3/2} \exp \left\{ -\frac{\lambda^2 x}{2} \right\}, \quad x > 0,$$

thus, the BG index is  $\beta = 1/2$ . See [43] for more details. The inverse Gaussian distribution has moments of every order finite and for every  $q \in \mathbb{R}$  we can express them as

$$\begin{aligned} E|X(1)|^q &= \int_0^\infty x^q f(x) dx = \frac{\delta}{\sqrt{2\pi}} e^{\delta\lambda} \left(\frac{2}{\lambda^2}\right)^{q-1/2} \int_0^\infty x^{q-3/2} \exp\left\{-x - \frac{\delta^2 \lambda^2}{4x}\right\} dx \\ &= \frac{\delta}{\sqrt{2\pi}} e^{\delta\lambda} \left(\frac{2}{\lambda^2}\right)^{q-1/2} K_{-q+\frac{1}{2}}(\delta\lambda) 2 \left(\frac{\delta\lambda}{2}\right)^{q-\frac{1}{2}} = \sqrt{\frac{2}{\pi}} e^{\delta\lambda} \delta^{q+\frac{1}{2}} \lambda^{-q+\frac{1}{2}} K_{-q+\frac{1}{2}}(\delta\lambda), \end{aligned}$$

where we have used [44, Eq. (10.32.10)] and  $K_\nu$  denotes the modified Bessel function of the second kind. This implies that

$$E|X(t)|^q = \sqrt{\frac{2}{\pi}} e^{t\delta\lambda} t^{q+\frac{1}{2}} \delta^{q+\frac{1}{2}} \lambda^{-q+\frac{1}{2}} K_{-q+\frac{1}{2}}(t\delta\lambda) \sim C t^{q+\frac{1}{2}} t^{-| -q+\frac{1}{2} |}, \quad \text{as } t \rightarrow 0,$$

since  $K_\nu(z) \sim \frac{1}{2}\Gamma(\nu)(\frac{1}{2}z)^{-\nu}$  for  $z > 0$  and  $K_{-\nu}(z) = K_\nu(z)$ . For any choice of  $\gamma > 0$  condition of Corollary 1 cannot be fulfilled, so the best we can say is that the lower bound is 0, in accordance with (25). Since negative order moments are finite, Corollary 2 yields the sharp upper bound on the spectrum. Indeed, given  $\gamma > 1/\beta = 2$  we have for  $q < 1/(2 - \gamma) < 0$

$$P(|X(t)| \leq Kt^\gamma) \leq \frac{E|X(t)|^q}{K^q t^{\gamma q}} \leq C t^{-q(\gamma-2)},$$

for  $t$  sufficiently small. It follows that the upper bound is 2 which is exactly the reciprocal of the BG index.

#### 4.2.2 Tempered Stable Subordinator

The positive tempered stable distribution is obtained by exponentially tilting the Lévy density of the  $\alpha$ -stable distribution,  $0 < \alpha < 1$ . The tempered stable subordinator is a Lévy process  $\{X(t)\}$  such that  $X(1)$  has a positive tempered stable distribution given by the cumulant function

$$\Phi(\theta) = \log E[e^{-\theta X(1)}] = \delta\lambda - \delta(\lambda^{1/\alpha} + 2\theta)^\alpha, \quad \theta \geq 0,$$

where  $\delta$  is the scale parameter of the stable distribution and  $\lambda$  is the tilt parameter. In this case the BG index is equal to  $\alpha$  (see [45] for more details). We use Corollary 2 for  $\gamma > \alpha$  to get

$$P(|X(t)| \leq Kt^\gamma) \leq eE\left[e^{-\frac{X(t)}{Kt^\gamma}}\right] = e^{1+t\Phi(K^{-1}t^{-\gamma})} = O(e^{-t^{1-\gamma/\alpha}}), \text{ as } t \rightarrow 0.$$

As this decays faster than any power of  $t$  as  $t \rightarrow 0$ , the upper bound follows.

### 4.3 Multiplicative Cascade

Although it is ambiguous what multifractality means, some models are usually studied in this sense. One of the first models of this kind is the multiplicative cascade. Cascades are actually measures, but can be used to construct non-negative increasing multifractal processes. The discrete cascades satisfy only discrete scaling invariance, in the sense that Definition 2 is valid only on a discrete grid of points. Another drawback of these processes is the nonstationarity of increments.

In [34], a class of measures has been constructed having continuous scaling invariance and called multifractal random measures, thus generalizing the earlier cascade models. We will refer to a process obtained from such measure simply as the cascade. Since this is a notable example of a theoretically well developed multifractal process, we analyze it in the view of the results of the preceding section. Furthermore, we consider only one cascade process, the log-normal cascade (LNC). One can use cascades as subordinators to BM to build more general models called log-infinitely divisible multifractal processes (see [34, 46] and the references therein).

The following properties hold for the log-normal cascade  $\{X(t)\}$  with parameter  $\lambda^2$  ([47]). First,  $\{X(t)\}$  satisfies Definition 1 with the random factor  $M(c) = ce^{2\Gamma_c}$  where  $\Gamma_c$  is Gaussian random variable and can therefore be considered as a true multifractal. The

moment scaling holds with

$$\tau_{LNC}(q) = q(1 + 2\lambda^2) - 2\lambda^2 q^2.$$

The increments of  $\{X(t)\}$  are heavy-tailed with tail index equal to  $1/(2\lambda^2)$  and moments of every negative order are finite provided  $\lambda^2 < 1/2$  (see [48, Proposition 5]). Although the asymptotic behaviour of the scaling function defined by (8) is unknown, there are results for the estimator defined by (9). Fixed domain asymptotic properties of this estimator for the multiplicative cascade have been established in [49] where it was shown that when  $j \rightarrow \infty$ , the estimator (9) tends a.s. to

$$\tau_{LNC}^\infty(q) = \begin{cases} h_0^- q, & \text{if } q \leq q_0^-, \\ q(1 + 2\lambda^2) - 2\lambda^2 q^2, & \text{if } q_0^- < q < q_0^+, \\ h_0^+ q, & \text{if } q \geq q_0^+, \end{cases} \quad (26)$$

where

$$q_0^+ = \inf\{q \geq 1 : q\tau'(q) - \tau(q) + 1 \leq 0\} = \frac{1}{\sqrt{2\lambda^2}}, \quad (27)$$

$$q_0^- = \sup\{q \leq 0 : q\tau'(q) - \tau(q) + 1 \leq 0\} = -\frac{1}{\sqrt{2\lambda^2}} \quad (28)$$

and  $h_0^+ = \tau'(q_0^+)$ ,  $h_0^- = \tau'(q_0^-)$ . Hence, the estimator (9) is consistent for a certain range of  $q$ , while outside this interval the so-called linearization effect happens. Similar results have been established in the mixed asymptotic framework [50]; see also [51] for a different method. The spectrum of the log-normal cascade is supported on the interval  $\left[1 + 2\lambda^2 - 2\sqrt{2\lambda^2}, 1 + 2\lambda^2 + 2\sqrt{2\lambda^2}\right]$ , given by

$$d_{LNC}(h) = \inf_{q \in (-\infty, 1/(2\lambda^2))} (hq - \tau_{LNC}(q) + 1) = 1 - \frac{(h - 1 - 2\lambda^2)^2}{8\lambda^2},$$

and the multifractal formalism holds [52].

The condition  $\tau(q) > 1$  of Proposition 1 yields  $q \in (1, 1/(2\lambda^2))$ . We then get that

$$H^- = 1 + 2\lambda^2 - 2\sqrt{2\lambda^2}.$$

This is exactly the left endpoint of the interval where the spectrum of the cascade is defined, in accordance with Proposition 1. This maximal lower bound is achieved for  $q = 1/\sqrt{2\lambda^2} = q_0^+$ . If  $q^-$  is the point at which maximal lower bound  $H^-$  is achieved, then

$$\left(\frac{\tau(q)}{q} - \frac{1}{q}\right)' = \frac{1}{q^2} (q\tau'(q) - \tau(q) + 1)$$

must be equal to 0 at  $q^-$ . This is exactly defined in (27). Although the range of finite moments is not relevant for computing  $H^-$  in this case, in general it can depend on  $\bar{q}$ .

Since all negative order moments are finite we get that

$$H^+ = \widetilde{H}^+ = 1 + 2\lambda^2 + 2\sqrt{2\lambda^2}$$

achieved for  $q = -1/\sqrt{2\lambda^2}$ . Thus again the bound from Proposition 2 is sharp giving the right endpoint of the interval where the spectrum is defined.

#### 4.4 Multifractal Random Walk

With this example we want to show that we may have  $H^+ \neq \widetilde{H}^+$  and that the definition of the scaling function needs to be adjusted to avoid infinite moments of negative order. The log-normal multifractal random walk (LNMRW) is a compound process  $X(t) = B(\theta(t))$  where  $B$  is a BM and  $\theta$  is the independent LNC (see [34]). The multifractal properties of this process are inherited from those of the underlying cascade. Indeed,  $\{X(t)\}$  satisfies Definition 1 with the random factor  $M(c) = c^{1/2} e^{\Gamma_c}$  where  $\Gamma_c$  is a Gaussian random

variable and the scaling function is given by

$$\tau_{LNMW}(q) = q \left( \frac{1}{2} + \lambda^2 \right) - \frac{\lambda^2}{2} q^2.$$

The range of finite moments is  $(-1, 1/\lambda^2)$  as explained in Subsec. 3.4. The spectrum is finite on the interval  $\left[ 1/2 + \lambda^2 - \sqrt{2\lambda^2}, 1/2 + \lambda^2 + \sqrt{2\lambda^2} \right]$  and given by

$$d_{LNMW}(h) = 1 - \frac{(h - 1/2 - \lambda^2)^2}{2\lambda^2}.$$

The random factor  $M(c)$  is the source of multifractality, has the same scaling function, but all negative order moments are finite. Thus we get

$$\begin{aligned} H^- &= 1/2 + \lambda^2 - \sqrt{2\lambda^2}, \\ H^+ &= \frac{3}{2} + \frac{3\lambda^2}{2}, \\ \widetilde{H}^+ &= 1/2 + \lambda^2 + \sqrt{2\lambda^2}. \end{aligned}$$

One can see that  $H^-$  and  $\widetilde{H}^+$  give the sharp bounds, while  $H^+$  is affected by the divergence of negative order moments. This shows that when the multifractal process has infinite negative order moments, one should specify scaling in terms of the random factor.

## 5 ROBUST VERSION OF THE PARTITION FUNCTION

In Sec. 3 using Corollary 3 we managed to avoid the problematic infinite moments of negative order and prove results like Theorem 4 and Corollary 5. When the scaling function (8) is estimated from the data, spurious concavity may appear for negative values of  $q$  due to the effect of diverging negative order moments. We use the idea of

Corollary 3 to develop a more robust version of the partition function.

Instead of using plain increments in the partition function (6), we can use the maximum of some fixed number  $m$  of the same length increments. This will make negative order moments finite for some reasonable range and prevent divergencies. The underlying idea also resembles the wavelet leaders method where leaders are formed as the maximum of the wavelet coefficients over some time scale (see [25]). Since  $m$  is fixed, this does not affect the true scaling. The same idea can be used for  $q > 0$  by an argument following from Corollary 1. It is important to stress that the estimation of the scaling function makes sense only if the underlying process is known to possess scaling property of the type (5).

Suppose  $\{X(t)\}$  has stationary increments and  $X(0) = 0$ . Divide the interval  $[0, T]$  into  $\lfloor T/(mt) \rfloor$  blocks each consisting of  $m$  increments of length  $t$  and define the modified partition function:

$$\tilde{S}_q(T, t) = \frac{1}{\lfloor T/(mt) \rfloor} \sum_{i=0}^{\lfloor T/(mt) \rfloor - 1} \left( \max_{l=1, \dots, m} |X(imt + lt) - X(imt + (l-1)t)| \right)^q. \quad (29)$$

One can see  $\tilde{S}_q(T, t)$  as a natural estimator of the moment in (21). Analogously we define the modified scaling function as in (8) by using  $\tilde{S}_q(n, t_i)$ :

$$\tilde{\tau}_{N,T}(q) = \frac{\sum_{i=1}^N \ln t_i \ln \tilde{S}_q(n, t_i) - \frac{1}{N} \sum_{i=1}^N \ln t_i \sum_{j=1}^N \ln \tilde{S}_q(n, t_j)}{\sum_{i=1}^N (\ln t_i)^2 - \frac{1}{N} \left( \sum_{i=1}^N \ln t_i \right)^2}. \quad (30)$$

Another possibility is to change the original definition only for  $q < 0$  although there is no much difference between two forms when  $q > 0$ .

To illustrate how this modification makes the scaling function more robust we present several examples comparing (8) and (30). We generate sample paths of several processes and estimate the scaling function by both methods. We also estimate the spectrum numerically using (11). The results are shown in Figures 1-4. Each figure shows the



estimated scaling functions and the estimated spectrum by using standard definition (8) and by using (30). We also added the plots of the scaling function that would yield the correct spectrum via multifractal formalism and the true spectrum of the process.

For the BM (Figure 1) and the  $\alpha$ -stable Lévy process (Figure 2) we generated sample paths of length 10000 and we used  $\alpha = 1$  for the latter. The LFSM (Figure 3) was generated using  $H = 0.9$  and  $\alpha = 1.2$  with path length 15784 (see [53] for details on the simulation algorithm used). Finally, the LNMRW of length 10000 was generated with  $\lambda^2 = 0.025$  (Figure 4). For each case we take  $m = 20$  in defining the modified partition function (29).

In all the examples considered, the modified scaling function is capable of yielding the correct spectrum of the process with the multifractal formalism. As opposed to the standard definition, it is unaffected by diverging negative order moments. Moreover, it captures the divergence of positive order moments which determines the shape of the spectrum.

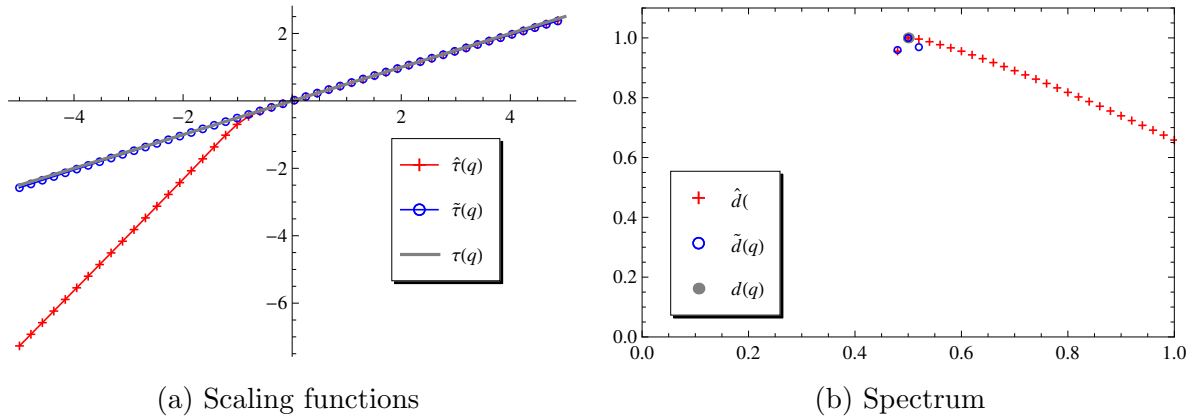


Figure 1: Brownian motion

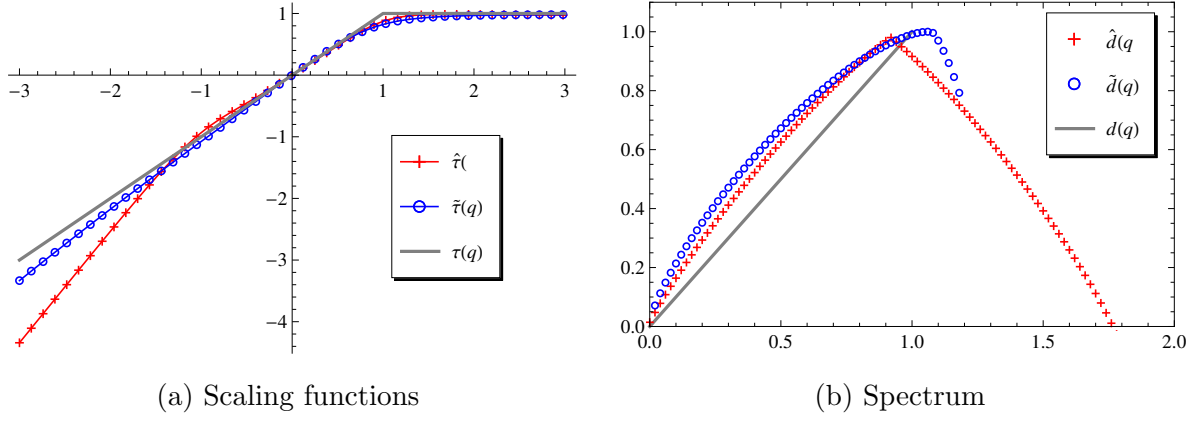


Figure 2: Stable Lévy motion  $\alpha = 1$

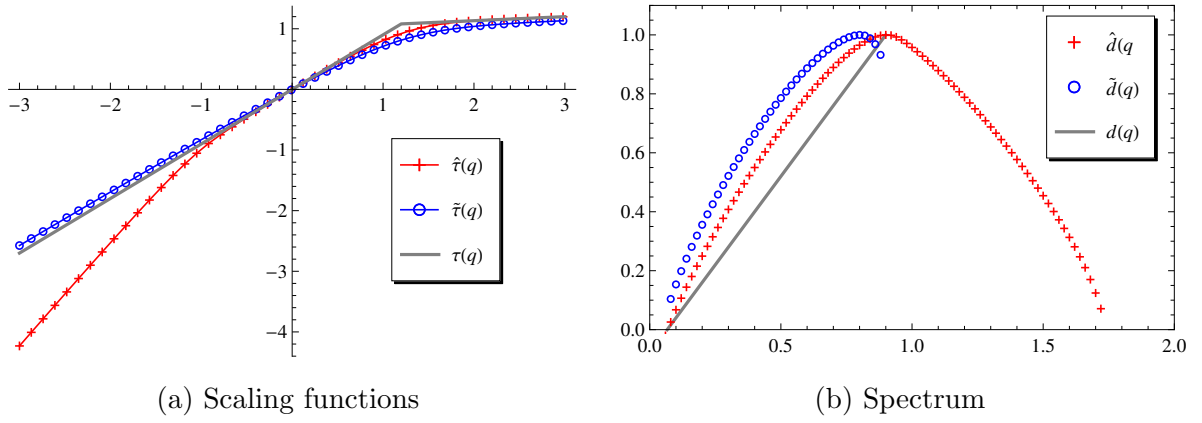


Figure 3: Linear fractional stable motion  $H = 0.9$ ,  $\alpha = 1.2$

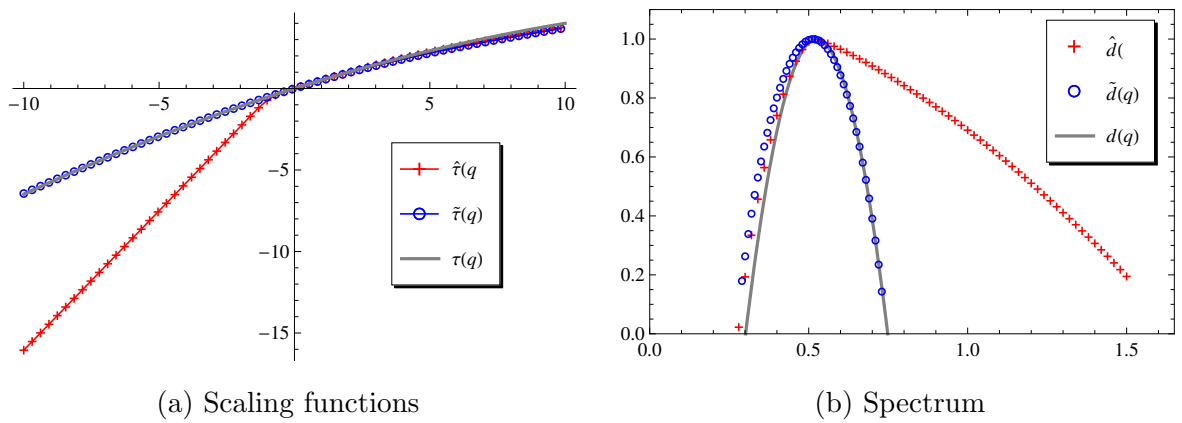


Figure 4: Log-normal multifractal random walk  $\lambda^2 = 0.025$

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## APPENDIX

We provide a brief overview of different classes of stochastic processes that are used along the paper.

A Lévy process is a process with stationary and independent increments starting from 0 and stochastically continuous at 0. Given an infinitely divisible distribution there exists a Lévy process such that  $X(1)$  has this distribution. Moreover, the characteristic function can be uniquely represented by the Lévy-Khintchine formula. See [41] and [45] for more details.

An  $\alpha$ -stable Lévy motion is a process such that  $X(1)$  has strictly stable distribution with stability index  $0 < \alpha < 2$ . In general, a random variable  $X$  has an  $\alpha$ -stable distribution with index of stability  $\alpha \in (0, 2)$ , scale parameter  $\sigma \in (0, \infty)$ , skewness parameter



$\beta \in [-1, 1]$  and shift parameter  $\mu \in \mathbb{R}$ , denoted by  $X \sim S_\alpha(\sigma, \beta, \mu)$ , if its characteristic function has the following form

$$E \exp\{i\zeta X\} = \begin{cases} \exp\left\{-\sigma^\alpha |\zeta|^\alpha \left(1 - i\beta \operatorname{sign}(\zeta) \tan \frac{\alpha\pi}{2} + i\zeta\mu\right)\right\}, & \text{if } \alpha \neq 1, \\ \exp\left\{-\sigma |\zeta| \left(1 - i\beta \frac{2}{\pi} \operatorname{sign}(\zeta) \ln |\zeta| + i\zeta\mu\right)\right\}, & \text{if } \alpha = 1, \end{cases} \quad \zeta \in \mathbb{R}.$$

The stable distribution is strictly stable if  $\mu = 0$  and  $\alpha \neq 1$ , or if  $\beta = 0$  and  $\alpha = 1$ . The stable Lévy motion is  $1/\alpha$ -sssi.

The linear fractional stable motion (LFSM) is an example of a process with heavy-tailed and dependent increments. It can be defined through the stochastic integral

$$X(t) = \frac{1}{C_{H,\alpha}} \int_{\mathbb{R}} \left( (t-u)_+^{H-1/\alpha} - (-u)_+^{H-1/\alpha} \right) dL_\alpha(u),$$

where  $\{L_\alpha\}$  is a strictly  $\alpha$ -stable Lévy process,  $\alpha \in (0, 2)$ ,  $0 < H < 1$  and  $(x)_+ = \max(x, 0)$ . The constant  $C_{H,\alpha}$  is chosen such that the scaling parameter of  $X(1)$  equals 1, i.e.

$$C_{H,\alpha} = \left( \int_{\mathbb{R}} \left| (1-u)_+^{H-1/\alpha} - (-u)_+^{H-1/\alpha} \right|^\alpha du \right)^{1/\alpha}.$$

It is then called standard LFSM. The LFSM is  $H$ -sssi. Setting  $\alpha = 2$  in the definition reduces the LFSM to the FBM. By analogy to this process, the case  $H > 1/\alpha$  is referred to as a long-range dependence and the case  $H < 1/\alpha$  as negative dependence. The parameter  $\alpha$  governs the tail behaviour of the marginal distributions implying, in particular, that  $E|X(t)|^q = \infty$  for  $q \geq \alpha$ . For more details see [54].

A Lévy process  $\{Y(t)\}$  such that  $Y(1) \sim S_\alpha(\sigma, 1, 0)$ ,  $0 < \alpha < 1$  is referred to as the stable subordinator. It is nondecreasing and  $1/\alpha$ -sssi. The inverse stable subordinator  $\{X(t)\}$  is defined as

$$X(t) = \inf \{s > 0 : Y(u) > t\}.$$

It is  $\alpha$ -ss with dependent, nonstationary increments and corresponds to a first passage time

of the stable subordinator strictly above level  $t$ . For more details see [55] and references therein.